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Charged lepton and neutrino oscillations

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Abstract. Problems long present in the conventional formalism employed for neutrino oscillations are discussed. We here develop a more satisfactory framework based on the Dirac equation and its propagators. When 4-momentum conservation is strictly enforced, there will be induced oscillations in space (but not between generations) for the charged leptons, e.g. μ and τ, produced in association with the neutrinos. The oscillations are computed explicitly for the pion decay $\pi \to \mu + \bar{\nu}$. Leptonic decays of the W are also briefly discussed.

1 Introduction

Neutrino oscillations have a long history beginning from the pioneering work of Pontecorvo and co-workers [1, 2]. The essential idea is very simple. If neutrinos have masses and if they mix, then as neutrinos propagate, the probability to find a given flavor neutrino would oscillate. However, it will be shown in the next section, that the conventional formalism, see for example [3–6], used to obtain this probability distribution is incorrect as it violates 4-momentum conservation. It is curious that for over 40 years, the well known and well confirmed Dirac equation has not been used for the purpose of describing neutrino oscillations. The Dirac equation does give the correct result. It also gives us a bonus in that a recoiling charged lepton (say, a μ or a τ) would also show oscillation (in space) in its decay. This cannot be avoided if 4-momentum conservation is strictly imposed. Needless to say, oscillations in a decaying charged lepton, say a muon, would constitute an excellent confirmation of neutrino oscillation.

Before discussing the details of how exactly charged lepton oscillations arise, it is perhaps worthwhile to comment on a general fallacy prevalent in some work in high energy physics. Over the past three years, while discussing our results with our colleagues both privately and in seminars, we have been struck by the fact that many believed "oscillations" to be a rather singular phenomenon to be associated only with neutral particles, such as the $K\bar{K}$, $B\bar{B}$ system or with massive neutrinos as very special cases. In fact, "oscillations" describe a normal quantum interference phenomenon, present for charged or neutral systems whenever there is a superposition of more than one amplitude. There has been some disbelief that muons can oscillate. But of course, muons can oscillate! Electron oscillations occur in such variety that there are industrial applications for low energy electron diffraction. Many physicists seem unaware that the famous CERN experiments [7] of the seventies measuring $(g-2)$ of the muon is based upon observing the frequency of oscillations in the muon decay probability induced by an imposed magnetic field. Schwinger showed [8] long ago that the muon mass is split in an external electromagnetic field provided $g \neq 2$. Thus, as a muon propagates, its spatial decay probability shows an interference pattern due to the superposition of two amplitudes. The frequency in the muon decay pattern is then directly proportional to $(g-2)$. While the muon decay pattern in $(g-2)$ experiments is sometimes thought to merely describe magnetic field induced spin precession, it is also true that $K\bar{K}$ oscillations may also be viewed as a formal "spin precession" inherent for quantum interference in two state systems.

The paper is organized as follows. In Sect. 2, we shall recall the standard treatment of neutrino oscillations (as found in textbooks and in the research literature) and show its inadequacy. In Sect. 3, we discuss the production of a muon and its neutrino from a pion decay. We shall exhibit their propagation in space time. We shall also exhibit how this double distribution oscillates by virtue of 4-momentum conservation. (A similar result has been discussed previously by us where a Λ produced in association with a K was found to oscillate). Pion and muon lifetime effects are discussed in Sect. 4. Relevant aspects of the muon spectra observed in previous pion decay experiments are covered in Sect. 5. In Sect. 6, are some concluding remarks with a brief comment on the τ spectra from a W decay. Finally, in an Appendix to this work, the $K\bar{K}$ system is reviewed (for completeness of presentation) using a formal "spin precession" picture. This picture brings out clearly why the notion of quantum interference in $K\bar{K}$ oscillations is closely analogous to the physical spin precession oscillations detected in muon $(g-2)$ experiments.

2 Conventional analysis

The conventional analysis of neutrino oscillation proceeds as follows. Neutrinos of various flavors (ν_l, where $l = e$, μ and τ) are related to the mass (m_a) eigenstates of neutrinos (ν_a, where $a = 1, 2, 3$) through the rotation matrix

$$|\nu_l> = \sum_a R_{la}|\nu_a>, \tag{2.1}$$

The amplitude to find a neutrino of flavor l' at time t, if a flavor l were produced at time 0, is then written as

$$\begin{aligned} <\nu_{l'}(t)|\nu_l(0)> &= <\nu_{l'}(0)|e^{-iHt}|\nu_l(0)> \\ &= \sum_a R^*_{l'a} R_{la} e^{-iE_a t}, \end{aligned} \tag{2.2}$$

where the energy of the ath neutrino is taken to be

$$E_a = \sqrt{\mathbf{q}^2 \ + \ m_a^2}, \tag{2.3}$$

with $\mathbf{q}$ assumed to be the common 3-momentum of the neutrinos.

The above expression showing neutrino oscillations is elegant, compact and wrong. We can easily see that it is wrong because it violates Lorentz invariance. (We can suspect that it is not quite right because the 3-momentum of the neutrinos cannot be the same in all Lorentz frames). Let us show it explicitly for the case of a pion of 4-momentum k_λ which decays into a charged lepton (say, a muon) of mass M_μ and a muon neutrino. In terms of the 4-momenta of the muon and the mass eigenstates of the neutrinos, overall 4-momentum conservation implies

$$k_\lambda = p(a)_\lambda + q(a)_\lambda, \ a = 1, 2, 3 \tag{2.4}$$

where the muon 4-momenta $p(a)_\lambda$ and the neutrino 4-momenta $q(a)_\lambda$ must be different for each of the three channels [9, 10, 11]. This contradicts (2.3) with the assumed common 3-momentum for the neutrinos.

Equation (2.4) not only lays bare the fallacy in the usual expression used for neutrino oscillations, but it also shows that the recoiling muon momenta are also different for each of the three channels. This implies that if we observe the muon produced in association with a neutrino at some spatial distance away from the pion decay, it will also show an interference pattern (i.e., "oscillate" in space) since it would be in a superposition of different 4-momenta [12]. This is our central result: muons would oscillate whenever they are produced in association with neutrinos which have masses and which mix.

A similar result, called Λ oscillations, was shown by us previously [13] where a Λ recoiling against a neutral kaon (a linear combination of K_L and K_S which have different masses) also shows spatial oscillations for exactly the same reasons.

Apart from the algebraic shortcoming, there is another serious problem with (2.2). An inner product implies an integration (or summation) over some variable. What is being integrated upon is not specified there. We may guess that there is a spatial integral, i.e., both the bra and the ket had a common position which is being summed. But that would be very odd for most neutrino oscillation analyses. Let us consider the case of solar neutrinos. At best we may know roughly where it was produced and where on earth it is detected. So the variable best specified is spatial and what one is really trying to measure is some oscillation in space. The reader will agree that one should not integrate over space if one is interested in spatial interference (or oscillation).

The above lacunae are rectified if we follow the standard procedure of following the particles from their production, to their propagation and where necessary their subsequent decays. In the next two sections we shall develop the complete formalism starting from the Dirac equation and its propagators, to obtain the precise formula for the double distribution of a charged lepton produced along with a neutrino. All this would be exhibited for arbitrary wave packets [14]. We shall also show that pion width does not alter the "oscillation" pattern, only the overall envelope.

3 Dirac equation and propagators

For definiteness and its obvious practical utility, let us consider a pion decay into a muon and a muon neutrino. All the algebraic steps would be explicitly written out lest these results appear obscure.

In the standard model, the action for a charged pion decay is given by

$$\begin{aligned} \mathcal{S}_{int} = \int (d^4x) (\frac{G_F F_\pi cos\theta_c}{\sqrt{2}}) \partial\phi_\pi(x) \\ \cdot\bar{\psi}(\mu, x)\gamma(1-\gamma_5)\nu_\mu(x), \end{aligned} \tag{3.1}$$

with G_F the Fermi coupling constant, F_π the pion decay constant, θ_c the Cabibbo angle, ϕ_π the pion wave function, $\psi(\mu, x)$ the muon wave function and ν_μ the neutrino wave function.

The muon wave function satisfies the Dirac equation with a source $\eta(\mu, x)$

$$(i\gamma \cdot \partial - M_\mu)\psi(\mu, x) = \eta(\mu, x), \tag{3.2}$$

where the muon source is computed from (3.1) to be

$$\begin{aligned} \eta(\mu, x) = \frac{\delta \mathcal{S}_{int}}{\delta\bar{\psi}(\mu, x)} = (\frac{G_F F_\pi cos\theta_c}{\sqrt{2}}) \partial\phi_\pi(x) \\ \cdot\gamma(1-\gamma_5)\nu_\mu(x), \end{aligned} \tag{3.3}$$

The propagation of a muon, i.e., the amplitude for finding a muon at a space-time point x, with a source at x' is given by

$$\psi(\mu, x) = \int (d^4x') S_F(\mu; x, x')\eta(\mu, x'), \tag{3.4}$$

where $S_F(\mu; x, x')$ is the Feynman-Stükelberg propagator for the muon

$$(i\gamma \cdot \partial - M_\mu)S_F(\mu; x, x') = \delta^4(x - x'). \tag{3.5}$$

It is more useful to rewrite (3.4) in the second order formalism in terms of the scalar muon propagator $D(\mu; x, x')$ defined through

$$S_F(\mu; x, x') = (-i\gamma \cdot \partial - M_\mu) D(\mu; x, x'), \tag{3.6}$$

so that (3.4) reads

$$\psi(\mu, x) = -\int (d^4x') D_F(\mu; x, x') \times (i\gamma \cdot \partial' + M_\mu)\eta(\mu, x'). \tag{3.7}$$

The scalar propagator $D(t, \mathbf{r}; M_\mu)$ for a mass M_μ has the well known following form in the "energy" representation

$$D(t, \mathbf{r}; M_\mu) = \int_{-\infty}^{+\infty} (\frac{dE}{2\pi})(\frac{e^{-iEt+ip(E)r}}{4\pi r}), \quad p(E) = \sqrt{E^2 - M_\mu^2}. \tag{3.8}$$

Let us now turn to the neutrinos. The neutrino propagator is a 3 by 3 matrix, being diagonal in the mass eigenstate basis. For each mass m_a, we have the scalar propagator $D(y, y'; m_a)$, of the identical form as above. Thus, we follow exactly the arguments given above to find that for a muon neutrino source $\eta(\nu_\mu, y')$ at space time point y', the amplitude to find a given neutrino flavor l at y is obtained through

$$\nu(l, y) = -\sum_a R_{al} R_{\mu a} \int (d^4y') D(y, y'; m_a) \times (i\gamma \cdot \partial' + m_a)\eta(\nu_\mu, y'). \tag{3.9}$$

Once again, the neutrino propagation far away from its production is most easily seen through the representation given in (3.8) but here for a mass m_a.

These expressions are sufficient to investigate all possible "oscillations" for the leptons. If we are interested in the "double distribution", i.e., suppose we wish to compute the amplitude for detecting both leptons: a muon at a spatial position $\mathbf{r}$ and a possibly different flavor neutrino l at spatial position $\mathbf{r}'$. For this we would require the source for both the muon and a neutrino which is given by

$$\eta^\alpha_\beta(\mu, x; \nu_\mu, y) = \frac{\delta^2 \mathcal{S}_{int}}{\delta\bar{\psi}_\alpha(\mu, x)\delta\psi^\beta(\nu_\mu, y)} = (\frac{G_F F_\pi cos\theta_c}{\sqrt{2}}) \int (d^4z)\delta^4(x - z)\delta^4(y - z) \times \partial^z_\lambda \phi_\pi(z)[\gamma^\lambda(1 - \gamma_5)]^\alpha_\beta \tag{3.10}$$

The amplitude for obtaining a muon at x and a neutrino of flavor l at y is given by

$$\chi^\alpha_\beta(\mu \ at \ x; \nu_l \ at \ y) = \sum_a R_{al} R_{\mu a} \int\int (d^4x')(d^4y') \times D(x, x'; M_\mu) D(y, y'; m_a) Q^\alpha_\beta(x', y'), \tag{3.11}$$

with

$$Q^\alpha_\beta(x', y') = [i\gamma \cdot \partial_{x'} + M_\mu]^\alpha_\rho [i\gamma \cdot \partial_{y'} + m_a]^\sigma_\beta \eta^\rho_\sigma(x', y'). \tag{3.12}$$

Employing the energy-space decomposition for the scalar propagators given in (3.8), we find

$$\chi^\alpha_\beta(\mu \ at \ x; \nu_l \ at \ y) = \sum_a R_{al} R_{\mu a} \int (d^4x')(d^4y') \times \int \frac{dE}{2\pi}\frac{dE'}{2\pi} e^{-iE(t_x - t_{x'}) - iE'(t_y - t_{y'})} \chi^\alpha_\beta(\mathbf{x}, E; \mathbf{y}, E'; a), \tag{3.13}$$

where

$$\chi^\alpha_\beta(\mathbf{x}, E; \mathbf{y}, E'; a) = \frac{exp\big(ip(\mu, E)|\mathbf{x} - \mathbf{x}'| + ip(\nu_a, E')|\mathbf{y} - \mathbf{y}'|\big)}{4\pi r r'} \times Q^\alpha_\beta(x', y'). \tag{3.14}$$

We are interested in the asymptotic limit of large distances $r = |\mathbf{x} - \mathbf{x}'|$ and $r' = |\mathbf{y} - \mathbf{y}'|$. The result in this limit gives us the energy conserving delta function and is proportional to the initial pion 3-momentum distribution (corresponding to a "wave packet")

$$r r' \chi^\alpha_\beta(\mathbf{x}, E; \mathbf{y}, E'; a) \to (\frac{-i\pi}{16\pi^2}) \times \delta(E_\pi - E - E') \frac{A_\pi(\mathbf{k} = \mathbf{p} + \mathbf{q})}{\sqrt{(\mathbf{k}^2 + M_\pi^2)}} t^\alpha_\beta, \tag{3.15}$$

where $A_\pi(\mathbf{k})$ is an arbitrary initial pion 3-momentum distribution defined through the pion wave function as

$$\phi_\pi(x) = \int \Big(\frac{d^3\mathbf{k}}{(2\pi)^3 2E_\pi(\mathbf{k})}\Big) e^{-ik\cdot x} A_\pi(\mathbf{k}), \quad E_\pi(\mathbf{k}) = \sqrt{\mathbf{k}^2 + M_\pi^2}, \tag{3.16}$$

satisfying the normalization condition

$$\int \frac{(d^3\mathbf{k})|A_\pi(\mathbf{k})|^2}{(2\pi)^3 2E_\pi(\mathbf{k})} = 1. \tag{3.17}$$

The spin algebra is all contained in t^α_β

$$t^\alpha_\beta = [(\gamma \cdot p + M_\mu)(\gamma \cdot p + \gamma \cdot q)(1 - \gamma_5)(\gamma \cdot q + m_a)]^\alpha_\beta. \tag{3.18}$$

The neutrino mass dependences are only important in the phases. Hence we ignore the other unimportant neutrino mass dependences to simplify the subsequent expressions. Under this approximation, and upon performing the spin polarization sums, the outgoing muon and neutrino (of flavor l) double distribution is found to be

$$\frac{d^6N(\mu \ at \ r; \nu_\mu \to \nu_l \ at \ r')}{(dr d\Omega)(dr' d\Omega')} \approx (\frac{G_F F_\pi cos\theta_c}{\sqrt{2}})^2 \times [\frac{M_\mu(M_\pi^2 - M_\mu^2)}{(8\pi)}]^2 |\mathcal{M}|^2, \tag{3.19}$$

where the matrix element $\mathcal{M}$ is given by

$$\mathcal{M} = \int_{-\infty}^{\infty} (\frac{dE}{2\pi}) \int_{-\infty}^{\infty} (\frac{dE'}{2\pi}) \frac{A_\pi(\mathbf{k} = \mathbf{p} + \mathbf{q})}{\sqrt{(\mathbf{k}^2 + M_\pi^2)}} \delta(E_\pi - E - E') \times \sum_a R_{\mu a} R_{al} e^{-i\vartheta_a}, \tag{3.20}$$

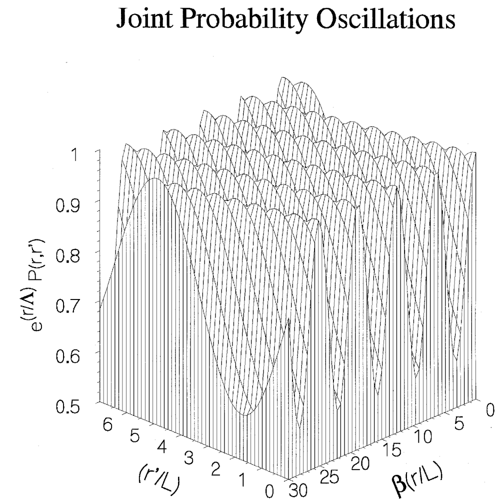

Fig. 1. Shown is a plot of $P(r,r')$ in (3.22) for two flavor mixing with a rotation angle $\theta = 20^o$. The joint probability distribution oscillates along the muon neutrino r' axis as well as along the muon r axis. The length scale is determined by the neutrino mass squared difference $L = |m_1^2 - m_2^2|/p$

in which the phase ϑ_a from each of the channel a reads

$$\vartheta_a = \vartheta(\mu,a)+\vartheta(\nu,a) = (\frac{M_\mu^2}{|\mathbf{p}(\mu,a)|})r+(\frac{m_a^2}{|\mathbf{q}(\nu,a)|})r. \quad (3.21)$$

The presence of the rotation matrices R and the dependence of the muon momentum $\mathbf{p}(\mu,a)$ on the neutrino type a, which occurs thanks to the exact 4-momentum conservation, explicitly proves the presence of oscillations in the muons recoiling against the various flavor neutrinos. This important result for the muon has been missed by previous workers since they had failed to employ the exact energy-momentum constraints.

To make explicit the probability $P = |\sum_a R_{\mu a} R_{al} e^{-i\vartheta_a}|^2$ via (3.20), let us consider a model in which only two flavors of neutrino mix. The electron and muon neutrinos are rotated through an angle θ. With r denoting the muon position and r' denoting the muon neutrino position, the joint probability distribution in the pion rest frame of the decay may be written as

$$P(r,r') = e^{-r/\Lambda}\Big(cos^4\theta + sin^4\theta + 2sin^2\theta cos^2\theta cos\phi(r,r')\Big), \quad (3.22a)$$

$$\phi(r,r') \approx \Big(\frac{m_1^2 - m_2^2}{p}\Big)(r - \beta r'), \quad (3.22b)$$

$$p = \frac{M_\pi^2 - M_\mu^2}{2M_\pi} \approx 29.8\ MeV, \quad \beta = \frac{M_\mu^2(M_\pi^2 + M_\mu^2)}{(M_\pi^2 - M_\mu^2)^2} \approx 4.95, \quad (3.22c)$$

$$\frac{1}{\Lambda} = \frac{\Gamma_\mu M_\mu}{p}. \quad (3.22d)$$

Here, Γ_μ^{-1} is the muon lifetime. Equations (3.22) are plotted for small mixing angle in Fig. 1.

The reader would please note that our result is valid for arbitrary wave packets, i.e., any initial momentum distribution of the pion specified by $A_\pi(\mathbf{k})$. It may not be superfluous to add that $A_\pi(\mathbf{k})$ is all that is generally specified in an experiment. It would be incorrect to specify both the 4-coordinates of the pion as well as its 4-momentum with arbitrary accuracy.

4 Pion and muon lifetime effects

As stated in the introduction, one may suspect that given the tiny mass differences expected for the various neutrino types, the effect obtained in the last section may not survive any "smearing" of the distribution due to the finite pion lifetime. Fortunately, such is not the case. Before proceeding to demonstrate it mathematically, let us first consider the following qualitative argument why the interference phases are untouched by the pion width. A width is in the imaginary part of a "mass operator" whereas interference phases are in its real part. Hence, widths in general reduce the overall probabilities but do not destroy the interference phases.

Now to a more quantitative demonstration. The pion propagator including its width Γ_π may be written as

$$D_\pi(x;\Gamma_\pi) = \int \frac{d^4k}{(2\pi)^4}\frac{e^{-ik\cdot x}}{M_\pi^2 - k^2 - iM_\pi\Gamma_\pi}. \quad (4.1)$$

It is convenient to use the Schwinger proper time representation [15], to convert the above into

$$D_\pi(x;\Gamma_\pi) = \int_0^\infty \Big(\frac{ds}{16\pi^2 s^2}\Big)e^{-i(M_\pi^2 - iM_\pi\Gamma_\pi)s - i(x^2/4s)}. \quad (4.2)$$

For large time-like arguments, $M_\pi\tau >> 1$, where the pion proper time $\tau = \sqrt{x^2}$, the above is evaluated using the steepest descent method to give

$$D_\pi(x;\Gamma_\pi) \to e^{-\Gamma_\pi\tau/2}D_\pi(x;0). \quad (4.3)$$

Thus, in the limit of interest to us, the width provides an overall multiplicative reduction factor for the entire amplitude leaving the oscillating phases intact in the "zero width" propagator. This justifies our neglect of the pion width.

The most natural way to include the muon width (that is, the muon life time effects) is to recast (3.21) in terms of proper times. Let us recall that the neutral K mesons, for which oscillations in strangeness have been experimentally verified, are indeed described in terms of proper times [16]. The muon has three possible proper times each corresponding to a neutrino mass eigenstate

$$M_\mu\tau_a = E(\mu,a)t - |\mathbf{p}(\mu,a)|r, \quad (4.4)$$

so that the muon part of the phase (for channel a), in a manifestly Lorentz invariant form reads

$$\vartheta(\mu,a;\Gamma_\mu) = (M_\mu - \frac{i}{2}\Gamma_\mu)\tau_a. \quad (4.5)$$

The above oscillation in the muon spatial probability part of the double distribution induced by the neutrino mixing, has an exact counterpart in processes where a particle is produced recoiling against a mixed K_L-K_S system. As we have shown previously [13], in the reaction $\pi^- + p \to \Lambda + K$, the recoiling Λ should display oscillations in its spatial development for the above reasons.

NOMAD, CHORUS and other neutrino baseline experiments, where the object is to discover signals for a "wrong" flavor neutrino far away, ought to be supplemented with coincident observations of spatial oscillations in the muon, as given by (3.20-21) and (4.4).

5 Muon spectra from previous pion decay experiments

In the past, several experiments have been performed on decays of stopped pions and pions in flight. For example, in [17–20] produced muons were observed in space. (For a theoretical analysis of this experiment, see [4]). The aim of these single distribution experiments was *not* to look for neutrinos. Only muons were detected. Such experiments might commonly be thought to involve "summing over the final neutrinos". The produced muons are subjected to a magnetic field, and one measures their radii accurately. Since the radius of curvature in a magnetic field depends linearly on the 3-momentum of the muon, and since these momenta depend upon the neutrino channel (see (2.4) and Sect. 3), the object of these experiments was to find (possibly several) peaks in the muon momentum corresponding to (different) neutrino masses. The experiments were unsuccessful in resolving these peaks in momenta since momentum differences (if any) are expected to be extremely tiny for small neutrino mass differences. For example, the difference in momentum $\Delta p \approx 10^{-10}\ eV/c$ for $\Delta m_\nu^2 \approx 10^{-2}(eV/c^2)^2$.

Precisely because the differences in momenta are small, and by direct momentum experiments impossible to resolve, quantum mechanics dictates to us that we try to observe these via its complementary variable, i.e., in space. Two (or, possibly three) very close peaks in momenta would appear in space as spread out spatial oscillations as in Fig. 1. For example, for neutrino mass difference in the above range, oscillations in the muon decay probability should be observable over a few decay lengths of the muon. Some further numerical results may be found in [12].

As already alluded to in the introduction, it is often forgotten that the famous $g-2$ experiments for the muon performed at CERN over two decades ago [7] are actually muon oscillation experiments. There the muon oscillates because, as Schwinger showed [8], in an external magnetic field the muon acquires two different "masses" if $(g \neq 2)$. But the induced energy splitting is so tiny for available magnetic fields, that it would be hopeless to measure it directly. In fact, in the CERN experiments, the induced spatial observations in the muon decays were measured (and so also for the ongoing Brookhaven experiments [21]). For a recent discussion of muon oscillations in $g-2$ experiments, see [22]. Exactly, what we are advocating here: to look for oscillations in the muon spatial decays due to the neutrino masses and their mixings.

6 Conclusions

We can summarize our results as follows. If 4-momentum conservation is strictly imposed in a pion decay into a muon and its neutrino, then if neutrinos have masses and they indeed do mix, the recoiling muons must also oscillate. Explicit expressions for such double distributions have been presented for an arbitrary initial pion wave packet. It has also been noted that direct measurements of the muon momenta (via its trajectory in a magnetic field) is not very fruitful for neutrino mass induced effects. A more promising avenue, and one which in an analogous context has been extremely successful [7, 22], is through its spatial oscillations manifested in the decays of the muon.

It is obvious that other decays can be treated similarly. For example, in a W decay into a τ and ν_τ, the τ decays in space would also show oscillations, albeit over much shorter distances. For a W decay at rest, the distance r corresponding to $\Gamma_\tau \tau = 1$ is of the order of 0.2 cm., for a τ momentum of the order of 40 GeV/c. The oscillations would be visible here only provided Δm_ν^2 is of the order of $10^{11}\ (eV/c^2)^2$.

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Appendix

Our purpose here is to review the notion of kaon oscillations using a modification of the Bloch equations for rotations in the kaon "quasi-spin". Our notion of $(K^0, \bar{K}^0)$ quasi-spin follows very closely the discussion of T. D. Lee [23]. A single particle kaon quantum state, as a function of proper time τ, may be written

$$|\psi(\tau)> = \begin{pmatrix} a_{K^0}(\tau) \\ a_{\bar{K}^0}(\tau) \end{pmatrix}, \qquad \text{(A1)}$$

where $a_{K^0}(\tau)$ and $a_{\bar{K}^0}(\tau)$ represent the amplitudes for the kaon to be, respectively, a K^0 and $\bar{K}^0$. The equation of motion for these amplitudes is given by

$$i\frac{\partial|\psi(\tau)>}{\partial\tau} = \mathcal{M}|\psi(\tau)> . \qquad \text{(A2)}$$

Under the assumption that $TCP = 1$, T.D. Lee [23] has shown that the complex mass matrix may be written in the form

$$\mathcal{M} = \Big(M - i(\frac{\Gamma}{2})\Big)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \Big(m - i(\frac{\gamma}{2})\Big)\begin{pmatrix} 0 & e^{\beta}e^{-i\theta} \\ e^{-\beta}e^{i\theta} & 0 \end{pmatrix}, \qquad \text{(A3)}$$

where the two real parameters β and θ are usually expressed in terms of a single complex parameter ϵ via

$$\epsilon = \Big(\frac{\sinh\beta - i\sin\theta}{\cosh\beta + \cos\theta}\Big). \tag{A4}$$

The strength of CP violation (or equivalently T violation if $TCP = 1$) is expressed by the finite size of $\beta \neq 0$. The conventional discussion of kaon oscillations makes use of mass eigenstates $\mathcal{M}|j>= (M_j - (i\Gamma_j/2))|j>$. Here $j = L$ for a "long" lived kaon and $j = S$ for a "short" lived kaon. The states are normalized to $<S|S>=<L|L>= 1$, but they are not orthonormal, i.e. $<S|L>= (\epsilon+\epsilon^*)/(1+|\epsilon|^2)$ due to $\beta \neq 0$ CP violation. States which are initially a superposition of long and short kaons, i.e. $|\psi>= c_S|S> +c_L|L>$, will exhibit oscillations at a frequency determined by the mass splitting $\Delta M = M_L - M_S$. There are many ways to appreciate such oscillations.

From the viewpoint of quasi-spins, we follow T.D. Lee [23] and introduce the quasi-spin Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{A5}$$

Equation (A3) has the form

$$\mathcal{M} = \Big(M - i(\frac{\Gamma}{2})\Big) + \frac{1}{2}\Big(\omega\cdot\sigma - i\nu\cdot\sigma\Big), \tag{A6}$$

where $\omega_3 = 0$ and $\nu_3 = 0$ as a consequence of $TCP = 1$. The density matrix, corresponding to the pure state (A1), may be defined

$$\begin{aligned}\rho(\tau) &= |\psi(\tau)><\psi(\tau)| \\ &= \begin{pmatrix} a^*_{K^0}(\tau)a_{K^0}(\tau) & a^*_{\bar K^0}(\tau)a_{K^0}(\tau) \\ a^*_{K^0}(\tau)a_{\bar K^0}(\tau) & a^*_{\bar K^0}(\tau)a_{\bar K^0}(\tau) \end{pmatrix}.\end{aligned} \tag{A7}$$

It obeys an equation of motion corresponding to (A2),

$$i\frac{\partial\rho(\tau)}{\partial\tau} = (\mathcal{M}\rho(\tau) - \rho(\tau)\mathcal{M}^\dagger). \tag{A8}$$

Seeking a solution of (A8) the form

$$\rho(\tau) = \frac{1}{2}\Big(P_0(\tau) + \mathbf{P}(\tau)\cdot\sigma\Big), \tag{A9}$$

yields from (A6),(A8) and (A9),

$$\frac{dP_0(\tau)}{d\tau} = -\Gamma P_0(\tau) - \nu\cdot\mathbf{P}(\tau), \tag{A10a}$$

$$\frac{d\mathbf{P}(\tau)}{d\tau} = \omega\times\mathbf{P}(\tau) - \Gamma\mathbf{P}(\tau) - \nu P_0(\tau). \tag{A10b}$$

Equations (A10) for the mean quasi-spin precession vector $\mathbf{P}(\tau)$ and the total kaon survival probability $P_0(\tau)$ are the central results of this Appendix. The quasi-spin Bloch Equations (A9) and (A10) reduce to the quantum Lee Equations (A1) and (A2) for the case where a kaon beam is described as a pure quantum state. The kaon "oscillations" are then described by the quasi-spin precession term $\omega\times\mathbf{P}$ in (A10b). Equations (A10) still hold true for kaon beams which are only partially coherent. Partial coherence is present, for example, in kaons produced by high energy beam dumps.

We have reviewed quantum oscillations for the kaons in a density matrix *quasi-spin precession* language in order to clarify the very close analogy between kaon oscillations, and muon oscillations. The muon spin precession enters into $(g-2)$ muon decay experiments, while the Kaon quasi-spin precession enters into kaon decay experiments.

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